Quantum correlations VI

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- *Quantum correlations.*

- In this lecture we will be concerned with the main topic of these lectures: *quantum correlations*.

- Firstly, an exposition of quantization of the principal measure of correlations, correlation coefficient will be given.

- Secondly, we will define and study the coefficient of quantum correlations.

- Finally, we will indicate why the techniques described in the previous lectures are indispensable for that purpose.

- Warning: existence of correlations in the quantum theory, likewise in the classical case, is not equivalent to the existence of causal relations.

- As the first step, we recall that the correlation coefficient for the classical case was given as:

$$C(X,Y) = \frac{E(XY) - E(X)E(Y)}{(E(X^2) - E(X)^2)^{\frac{1}{2}}(E(Y^2) - E(Y)^2)^{\frac{1}{2}}}$$

- We rewrite this definition in the quantum context.

- Let a C^*-algebra $\mathfrak{A}$ be a specific algebra of observables (as in Rule 1), φ a state on $\mathfrak{A}$ (as in Rule 3), and $A, A' \in \mathfrak{A}$ be observables.

- Further, we replace the classical expectation value $E(X)$ by the quantum one $\varphi(A) \equiv \langle A \rangle$.

- We note (we advise to verify these formulas)

$$\left\langle (A - \langle A \rangle)^2 \right\rangle = \langle A^2 \rangle - \langle A \rangle^2 \tag{1}$$

and

$$\langle (A - \langle A \rangle)(A' - \langle A' \rangle) \rangle = \langle AA' \rangle - \langle A \rangle \langle A' \rangle \tag{2}$$

- Consequently, in the quantum context, one can write

$$C_q(A, A') = \frac{\langle (A - \langle A \rangle)(A' - \langle A' \rangle) \rangle}{\left\langle (A - \langle A \rangle)^2 \right\rangle^{\frac{1}{2}} \left\langle (A' - \langle A' \rangle)^2 \right\rangle^{\frac{1}{2}}} \tag{3}$$

- This form of correlation coefficient agrees with that given in Omnés book.

- Further, we note that an application of Schwarz inequality shows that $C_q(A, A') \in [-1, +1]$, so $C_q(A, A')$ is normalized.

- Again, it is advised to verify the above statement!

- In particular, one can speak about "quantum positive correlations" etc.

- As the second step we wish to show that the correlation coefficient, $C_q(A, A')$, can recognize the "very entangled" states.

- Following Omnés one has:

Example 1. – *We consider the composite system such that its algebra of observables is given by $B(\mathcal{H}) \otimes B(\mathcal{H})$ and we take a state φ of the form $\varphi(\cdot) = \mathrm{Tr}(\varrho\ \cdot)$, where ϱ is a density matrix (on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$).*

– *Let us select two observables of the form $A = 0 \cdot P_{e_0} + 1 \cdot P_{e_1}$ and the same for A', where P_e stands for the orthogonal projector on the vector e; to shorten notation we write A instead of $A \otimes \mathbb{1}$ and A' instead of $\mathbb{1} \otimes A'$.*

– *We want to find a special state which gives maximal value of C_q.*

– *Observe that the condition $C_q = 1$ gives:*

–

$$0 = \left(\langle A^2\rangle - 2\langle A\rangle^2 + \langle A\rangle^2\right)\left(\langle (A')^2\rangle - 2\langle A'\rangle^2 + \langle A'\rangle^2\right) - \left[\langle AA'\rangle - 2\langle A\rangle\langle A'\rangle + \langle A\rangle\langle A'\rangle\right]^2 \quad (4)$$

– *or equivalently for our choice of* A *and* A' ($A^2 = A$, *etc*)

$$0 = \langle A\rangle\langle A'\rangle\left[1 - \langle A\rangle - \langle A'\rangle + 2\langle AA'\rangle\right] - \langle AA'\rangle^2 \quad (5)$$

– *Let us adopt the following convention:* $\rho_{ij',kl'} = \langle ij'|\rho|kl'\rangle$.
– *Assuming additionally that* $dim\mathcal{H} = 2$, *so considering two dimensional case, one has*
– $\mathrm{Tr}\rho_\varphi AA' = \rho_{11,11}$, $\mathrm{Tr}\rho_\varphi \mathbb{1}\otimes A' = (\rho_{11,11} + \rho_{01,01})$, $\mathrm{Tr}\rho_\varphi A\otimes\mathbb{1} = (\rho_{11,11} + \rho_{10,10})$.

– *The formula (5) can be rewritten as*

$$0 = (\rho_{11,11} + \rho_{10,10})\,(\rho_{11,11} + \rho_{01,01})\,(1 - \rho_{10,10} - \rho_{01,01}) - \rho^2_{11,11} \tag{6}$$

– *One can define maximally entangled state by*

$$\Psi = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \equiv \frac{1}{\sqrt{2}}(e_1 \otimes e_0 - e_0 \otimes e_1). \tag{7}$$

We put $\rho_\Psi = |\Psi\rangle\langle\Psi|$. *Then,* $(\rho_\Psi)_{11,11} = 0 = (\rho_\Psi)_{00,00}$, $(\rho_\Psi)_{01,01} = \frac{1}{2}$, $(\rho_\Psi)_{10,10} = \frac{1}{2}$. *Obviously, (6) is fulfilled for the state* ρ_Ψ.

– *Thus, the state* ρ_Ψ*, where* Ψ *is a maximally entangled vector, gives an example of maximal correlation coefficient for the observables* A *and* A'*, i.e.* $C_q(A, A') = 1$.

Example 2. *Let ω be a separable state on $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$,*

$$\omega(\cdot) = \sum_{i=1}^{n} \lambda_i (\omega_i^1 \otimes \omega_i^2)(\cdot),$$

where ω_i^k, $k = 1, 2$, $i = 1, 2, ..., n$ are states in $\mathfrak{S}_{\mathfrak{A}_k}$. It is a simple matter to check that, in general,

$$\omega(A_1 \otimes \mathbb{1} \cdot \mathbb{1} \otimes A_2) \neq \omega(A_1 \otimes \mathbb{1})\omega(\mathbb{1} \otimes A_2)$$

for $A_k \in \mathfrak{A}_k$.

Therefore, the state ω contains some correlations. However, as the state ω is separable one, these correlations are considered to be of classical nature only!

- The straightforward quantization of the correlation coefficient gives a device for finding the size of correlations.
- BUT, the coefficient C_q is not able to distinguish correlations of quantum nature from that of classical nature.
- Thus, a new measure of quantum correlation should be introduced.
- This will be done by defining the coefficient of quantum correlations.
- The basic idea to define "pure" quantum correlations is to "subtract" classical correlations.
- In other words we will look for the best approximation of a given state ω by separable states.
- However, a given state ω, in general, can possess various decompositions.

- Thus, to carry out the analysis of such approximations we should use the decomposition theory, described in fourth lecture.

- Now, we will proceed to coefficient of (quantum) correlations for a quantum composite system specified by $(\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{S}_{\mathfrak{A}})$, where $\mathfrak{A}_i$ are C^*-algebras.

- Thus we will consider C^*-algebra case.

- We begin with the definition of the restriction maps

$$(r_1\omega)(A) = \omega(A \otimes \mathbb{1}) \tag{8}$$

$$(r_2\omega)(B) = \omega(\mathbb{1} \otimes B), \tag{9}$$

where $\omega \in \mathfrak{S}_{\mathfrak{A}}$, $A \in \mathfrak{A}_1$, and $B \in \mathfrak{A}_2$.

- Clearly, $r_i : \mathfrak{S}_{\mathfrak{A}} \to \mathfrak{S}_{\mathfrak{A}_i}$ and the restriction map r_i is continuous (in weak-$*$ topology), $i = 1, 2$.

- Let us take a measure μ on $\mathfrak{S}_{\mathfrak{A}}$.

- Define

$$\mu_i(F_i) = \mu(r_i^{-1}(F_i)) \tag{10}$$

for $i = 1, 2$, where F_i is a Borel subset in $\mathfrak{S}_{\mathfrak{A}_i}$.

- It is easy to check that the formula (10) provides the well defined measures μ_i on $\mathfrak{S}_{\mathfrak{A}_i}$, $i = 1, 2$.

- Having two measures μ_1, μ_2 on $\mathfrak{S}_1$, and $\mathfrak{S}_2$ respectively, we want to "produce" a new measure $\boxtimes\mu$ on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$.

- To this end, firstly, let us consider the case of finitely supported probability measure μ:

$$\mu = \sum_{i=1}^{N} \lambda_i \delta_{\rho_i} \tag{11}$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{N} \lambda_i = 1$, and δ_{ρ_i} denotes the Dirac's measure.

- We define

$$\mu_1 = \sum_{i=1}^{N} \lambda_i \delta_{r_1 \rho_i} \tag{12}$$

and

$$\mu_2 = \sum_{i=1}^{N} \lambda_i \delta_{r_2 \rho_i}. \tag{13}$$

- Then

$$\boxtimes\mu = \sum_{i=1}^{N} \lambda_i \delta_{r_1 \rho_i} \times \delta_{r_2 \rho_i} \tag{14}$$

provides a well defined measure on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$.

- Here $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$ is understood as a measure space obtained as a product of two measure spaces $\mathfrak{S}_{\mathfrak{A}_1}$ and $\mathfrak{S}_{\mathfrak{A}_2}$.

- A measure structure on $\mathfrak{S}_{\mathfrak{A}_i}$ is defined as the Borel structure determined by the corresponding weak-$*$ topology on $\mathfrak{S}_{\mathfrak{A}_i}$, $i = 1, 2$.

- An arbitrary fixed decomposition of a state $\omega \in \mathfrak{S}_{\mathfrak{A}}$ corresponds to a measure μ such that $\omega = \int_{\mathfrak{S}} \nu d\mu(\nu)$.

- As there are, in general, many decompositions (it was pointed out in the fourth and fifth lectures) we will be interested in measures from the following set

$$M_\omega(\mathfrak{S}_{\mathfrak{A}}) \equiv M_\omega = \{\mu : \omega = \int_{\mathfrak{S}} \nu d\mu(\nu)\},$$

i.e. the set of all Radon probability measures on $\mathfrak{S}_{\mathfrak{A}}$ with the fixed barycenter ω.

- Take an arbitrary measure μ from M_ω. There exists a net of discrete measures (having a finite support) μ_k such that $\mu_k \to \mu$, and the convergence is understood in the weak-* topology on $\mathfrak{S}_{\mathfrak{A}}$.

- Defining μ_1^k (μ_2^k) analogously as μ_1 (μ_2 respectively; cf equations (12), (13)), one has $\mu_1^k \longrightarrow \mu_1$ and $\mu_2^k \longrightarrow \mu_2$, where again the convergence is taken in the weak-* topology on $\mathfrak{S}_{\mathfrak{A}_1}$ ($\mathfrak{S}_{\mathfrak{A}_2}$ respectively).

- Then define, for each k, $\boxtimes\mu^k$ as it was done in (14).

- We can verify that $\{\boxtimes\mu^k\}$ is convergent (in weak $*$-topology) to a measure on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$.

- Consequently, taking the weak-* limit we arrive at the measure $\boxtimes\mu$ on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. It follows easily that $\boxtimes\mu$ does not depend on the chosen approximation procedure.

- Now, we are in position to give the definition of the coefficient of quantum correlations, $d(\omega, A_1, A_2) \equiv d(\omega, A)$, where $A_i \in \mathfrak{A}_i$.

- **Definition 3.** *Let a quantum composite system* $(\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{S}_{\mathfrak{A}})$ *be given. Take a* $\omega \in \mathfrak{S}_{\mathfrak{A}}$. *We define the coefficient of quantum correlations as*

$$d(\omega, A) = \inf_{\mu \in M_\omega(\mathfrak{S}_{\mathfrak{A}})} \left| \int_{\mathfrak{S}_{\mathfrak{A}}} \xi(A) d\mu(\xi) - \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \xi(A)(d \boxtimes \mu)(\xi) \right| \quad (15)$$

- The formula (15) is a "measure" of extra non classical type of correlations.

- Namely, following the strategy of Kadison-Ringrose example, the example discussed in the fifth lecture, an evaluation of a distance between the given state ω and the set of approximative separable states is done.

- It is a simple matter to see that $d(\omega, A)$ is equal to 0 if the state ω is a separable one.
- The converse statement is much less obvious.
- However, we are able to prove it.
- Namely:

Theorem 4. *Let $\mathfrak{A}$ be the tensor product of two C*-algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$. Then state $\omega \in \mathfrak{S}_{\mathfrak{A}}$ is separable if and only if $d(\omega, A) = 0$ for all $A \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$*

- The basic idea of the proof of the statement that $d(\omega, A) = 0$ implies separability of ω relies on the study of continuity properties of the function

$$M_\omega(\mathfrak{S}_\mathfrak{A}) \ni \mu \mapsto \int_{\mathfrak{S}_\mathfrak{A}} \xi(A) d\mu(\xi) - \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \xi(A)(d \boxtimes \mu)(\xi) \qquad (16)$$

- The proof falls naturally into few steps.

1. $M_\omega(\mathfrak{S}_\mathfrak{A})$ is a compact set.
2. The mapping $M_\omega(\mathfrak{S}_\mathfrak{A}) \ni \mu \mapsto \boxtimes\mu \in M^+(\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2})$ is weakly continuous.
3. The continuity proved in the second step implies that the function (16) is a real valued, continuous function defined on a compact space.
4. Hence, by Weierstrass theorem, infimum is attainable.

Therefore, the condition $d(\omega, A) = 0$ means that

$$\omega(A) = \int_{\mathfrak{S}_{\mathfrak{A}}} \xi(A) d\mu_0(\xi) = \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \xi(A) d \boxtimes \mu_0(\xi), \qquad (17)$$

for all $A = A_1 \otimes A_2$.

But, this means the separability of ω!

- Theorem 4 may be summarized by saying that any separable state contains "classical" correlations only.

- Therefore, **an entangled state contains "non-classical" (or pure quantum) correlations**.

- To comment the question of separability of normal states we have two remarks:

1. $(indirect\ way)$

   As we have considered C^*-algebra case, taking a normal state $\varphi \in \mathfrak{S}^n_{\mathfrak{M}} \equiv \mathfrak{S}_{\mathfrak{M}} \cap \mathfrak{M}_* \subset \mathfrak{S}_{\mathfrak{M}}$, we can apply Theorem 4 for its analysis.

   If $d(\varphi, A) = 0$ we are getting a "separable" decomposition of φ.

   However, still one must check whether components of the decomposition are normal or not. In other words, one must examine whether the measure providing the given decomposition is supported by $\mathfrak{S}^n_{\mathfrak{M}}$.

   It is worth pointing out that the lecture fourth provides examples of measures being supported by $Ext(\mathfrak{S}^n_{\mathfrak{M}})$ (if additionally the condition SC is satisfied).

2. (*a possibility for a direct way*)

One can try to modify the results obtained for C^*-algebra case to that which are relevant for W^*-algebra case.

However, there are two essential differences.

The first one: the closure of convex hull should be carried out with respect to the operator space projective norm topology.

The second difference leads to a great problem.

Namely $\mathfrak{S}^n_{\mathfrak{M}}$ is compact, in general, with respect to another topology than that which gives compactness of $\mathfrak{S}_{\mathfrak{M}}$.

To illustrate this let us consider $\mathfrak{M} = B(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional Hilbert space. Then $\mathfrak{S}^n_{B(\mathcal{H})}$ is a compact subset of $\mathcal{F}_{\mathcal{T}}(\mathcal{H})$ when it is equipped with $\sigma(\mathcal{F}_T(\mathcal{H}), \mathcal{F}_C(\mathcal{H}))$-topology.

$\mathfrak{S}_{B(\mathcal{H})}$ is compact with respect to $\sigma(B(\mathcal{H})^*, B(\mathcal{H}))$-topology.

Moreover, although the restriction $(r\omega)(A) = \omega(A \otimes \mathbb{1})$, where $\omega \in (B(\mathcal{H} \otimes B(\mathcal{H}))^*$ is also well defined for a density matrix (it is given by the partial trace) the restriction r is not, in general, $\sigma(\mathcal{F}_T(\mathcal{H} \otimes \mathcal{H}), \mathcal{F}_C(\mathcal{H} \otimes \mathcal{H}))$ – $\sigma(\mathcal{F}_T(\mathcal{H}), \mathcal{F}_C(\mathcal{H}))$ continuous.

As the continuity of the restriction map r was crucial, the C^*-algebra case can not be straightforwardly modified.